Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech
Journal of Applied Mathematics and Mechanics 69 (2005) 27–37

A NEW TRACKING ALGORITHM FOR CERTAIN MECHANICAL SYSTEMS†

M. S. YEFREMOV, A. Ye. POLYAKOV and V. V. STRYGIN

Voronerzh

e-mail: mefremov@relex.ru

(*Received* 10 *April* 2003)

A new algorithm is proposed for synthesizing a delayed on-off control for tracking slow motions of a certain class of mechanical systems. The conditions for a control to exist are found and examples are given of the solution of the tracking problem for inverted and double inverted pendulums.

In systems of variable structure [1] the control function is generally discontinuous. The sliding mode method [2] is an important tool for constructing robust controls. Publications presenting the ideas which later became the basis of the sliding mode control method first appeared in the middle of the last century [3]. Sliding mode control methods have been widely used in control problems for various systems: electromechanical systems [4], pendulums [5], and many others.

To overcome difficulties due to the presence of delay in the control, a method employing what are known as ε-stabilizing solutions [6] will be proposed below.

There is a considerable literature on the subject of the control of electromechanical systems. Existence conditions have been established for motions of robot manipulators in the decomposition mode, taking into account the dynamics of the slave organs; the form of control laws has been found and the set of motions of the manipulator constructed [7]. Examples have been presented of the stabilization of the programmed manifold of a manipulator on a moving base and the stabilization of the programmed orientation of a pursuing body [8]. Controlled plane motions of a two-link mechanism along a horizontal plane have considered [9]. It has been proved [10] that in the problem of controlling an n-link manipulator having a singular mode of order $2m$, the optimal paths reach the singular mode in a finite time, and the optimal control has an infinite number of switchings. Algorithms have been proposed for controlling an asynchronous electric drive, based on the method of separation of motions, in problems of control and identification; tracking based on the speed and angle of rotation of the motor shaft has been considered [11].

In the development of these result, a new algorithm will be presented below, for synthesising an on-off control for mechanical systems, taking into account the presence in the control of a time delay; this has not been done in previous treatments [7–11]. The control is assumed to be of the on-off type, as in [7, 10], which is dictated by its ease of application. The synthesis of a control with delay for a mechanical system is based on linearizing the initial system [12] in the neighbourhood of the tracked path and subsequently decoupling the system of second-order equations as two subsystems of first-order equations with the desired properties: a stable system, and an unstable system for which a control is constructed and whose output is the input for the stable system. To decouple the system, one uses the method described in [13], involving the solution of a quadratic matrix equation. Algorithms constructing a control for a mechanical system are presented in both vector and scalar cases. The quality of the control is verified by the results of numerical simulation, in the scalar case – of the equation of an inverted pendulum, and in the vector case – of the equation of a double inverted pendulum.

†*Prikl. Mat. Mekh.* Vol. 69, No. 1, pp. 30–41, 2005.

0021–8928/$—see front matter.
doi: 10.1016/j.jappmathmech.2005.01.003

1. STATEMENT OF THE TRACKING PROBLEM IN THE GENERAL CASE

Consider a system of the form

$$F(\mathbf{x}(t), \dot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t)) = \mathbf{u}(t-h) \tag{1.1}$$

where $F: R^{3n} \to R^n$ is a smooth mapping, $\mathbf{x}(t)$ is an n-dimensional vector-valued function, $\mathbf{u} \in R^n$ is a vector control, R^n is Euclidean n-space, and $h > 0$ is a time delay.

System (1.1) will be considered together with an initial function

$$\mathbf{x}(t) = \mathbf{\Phi}(t) \quad \text{for} \quad t \in [-h, 0]; \quad \mathbf{\Phi}(t) = (\phi_1(t), \ldots, \phi_n(t))^T$$

To describe the class of feedback controls u we introduce a set Q of pairs of mappings $S: R^n \to R^k$, $G: R^k \to R^n$. We will seek feedback controls in the form

$$\begin{aligned} &\mathbf{u}(t-h) = G(\operatorname{sign} S_1(\mathbf{x}(t-h)), \ldots, \operatorname{sign} S_k(\mathbf{x}(t-h))), \quad h > 0 \\ &(S, G) \in Q, \quad S = (S_1, \ldots, S_k) \end{aligned} \tag{1.2}$$

Let us assume that the system must track some object. To describe the tracked path, we introduce a mapping $\mathbf{x}^*(t): R \to R^n$, which is the state vector of the tracked object. The tracking problem is to find a control $\mathbf{u}(t-h)$ guaranteeing that the deviation of the system (1.1) from the tracked path at any instant of time will be small in some sense.

Definition. We shall say that a solution of system (1.1) tracks $\mathbf{x}^*(t)$ if, for fixed $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon)$, a number $k \le n$, and a pair of mappings $(S, G) \in Q$ such that, for an initial function $\mathbf{\Phi}(t)$ satisfying the condition $\|\mathbf{\Phi}(0) - \mathbf{x}^*(0)\| < \delta$, the solution of the system

$$\begin{aligned} &F(\mathbf{x}(t), \dot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t)) = G(\operatorname{sign} S_1(\mathbf{x}(t-h)), \ldots, \operatorname{sign} S_k(\mathbf{x}(t-h))) \\ &\mathbf{x}(t) = \mathbf{\Phi}(t), \quad -h \le t \le 0 \end{aligned}$$

satisfies the inequality

$$\|\mathbf{x}(t) - \mathbf{x}^*(t)\| < \varepsilon, \quad \forall t \ge 0$$

The norms of a constant matrix A and constant vector $\mathbf{b}$ are defined as

$$\|A\| = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|; \quad \|\mathbf{b}\| = \max_{1 \le i \le n} |b_i|$$

For variable matrices $A(t)$ and vectors $\mathbf{b}(t)$ the corresponding norms will be

$$\|A\| = \max_{t \ge 0} \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|; \quad \|\mathbf{b}\| = \max_{t \ge 0} \max_{1 \le i \le n} |b_i|$$

2. TRACKING PROBLEM FOR A SECOND-ORDER SCALAR SYSTEM

Statement of the problem. Consider the scalar second-order differential equation

$$\ddot{x} = f(x, \dot{x}) + u(t-h), \quad x \in R \tag{2.1}$$

where $f: R^2 \to R$ is a twice continuously differentiable function and u is a delayed control. Let $\phi(t)$ be the initial function and $x^*(t)$ the tracked path. We have to indicate the constraints imposed on the system and the tracked object, and also to select a control in the form of (1.2), dependent on a fixed $\varepsilon > 0$ and the parameter values, in such a way that the system will follow the path $x^*(t)$ with a tracking error not exceeding ε.

Fundamental theorem. We transform system (2.1). Let $y(t) = x(t) - x^*(t)$ denote the tracking error. In variables y, system (2.1) will have the form

$$\ddot{y} = f(y + x^*, \dot{y} + \dot{x}^*) - \ddot{x}^* + u(t-h), \quad y \in R \tag{2.2}$$

Linearizing the latter system, we have

$$Ly \equiv \ddot{y}(t) - a^*(t)\dot{y}(t) - b^*(t)y(t) = f^*(t) + g(y, \dot{y}) + u(t-h) \tag{2.3}$$

where

$$a^* = \partial f(x^*, \dot{x}^*)/\partial \dot{y}, \quad b^* = \partial f(x^*, \dot{x}^*)/\partial y, \quad f^*(t) = f(x^*(t), \dot{x}^*(t)) - \ddot{x}^*(t)$$

$g(y, \dot{y}) = o(y^2, \dot{y}^2)$ is the remainder term of the expansion. It is well known that a positive constant N_1 exists such that, if $|y|^2 + |\dot{y}|^2 < \nu$ ($\nu > 0$), the following inequality holds

$$|g(y, \dot{y})| \le N_1(|y|^2 + |\dot{y}|^2) \tag{2.4}$$

For fixed t, the eigenvalues of the operator Ly are

$$\lambda_{1,2}(t) = [a^*(t) \pm \sqrt{(a^*(t))^2 + 4b^*(t)}]/2, \quad 0 \le t \le +\infty$$

Let us assume that the eigenvalues are simple and real, and

$$\|\lambda_1(t)\| < \lambda_{\max} < +\infty, \quad \lambda_2(t) \le -q < 0, \quad \forall t \ge 0 \tag{2.5}$$

We shall consider the case of slow motions x^*, which guarantees that $d\lambda_i(t)/dt$ ($i = 1, 2$) will be small. Put $c_0 = 1 + 1/q$.

Lemma 1. If for given $\varepsilon > 0$

$$|\dot{y}(t) - \lambda_2(t)y(t)| < \varepsilon, \quad t \in [0, T] \tag{2.6}$$

then

$$|g(y(t), \dot{y}(t))| < N\varepsilon^2, \quad t \in [0, T]; \quad N = N_1(c_0^2 + (1 + \lambda_{\max}c_0)^2) \tag{2.7}$$

Proof. We may infer from relation (2.6), using the reverse triangle inequality, that

$$|\dot{y}| < \varepsilon + \lambda_{\max}|y| \tag{2.8}$$

Estimating the solution of the system

$$\dot{y}(t) - \lambda_2(t)y(t) = z(t); \quad y(0) = y_0 = \phi(0) - x^*(0) \tag{2.9}$$

using the second inequality of (2.5), we obtain

$$|y| < e^{-qt}|y_0| + \int_0^t e^{-q(t-s)}|z(s)|ds < \left(e^{-qt}\frac{|y_0|}{\varepsilon} + \frac{1 - e^{-qt}}{q}\right)\varepsilon < \left(1 + \frac{1}{q}\right)\varepsilon = c_0\varepsilon \tag{2.10}$$

Estimate (2.7) follows from estimates (2.4), (2.8) and (2.10).

We now proceed directly to the construction of the control. Since $d\lambda_2(t)/dt$ is small, the left-hand side of Eq. (2.3) may be written as $(d/dt - \lambda_1(t))(d/dt - \lambda_2(t))y(t)$. Substituting

$$z(t) = \dot{y}(t) - \lambda_2(t)y(t)$$

we obtain the following problem

$$\begin{aligned} \dot{z}(t) &= \lambda_1(t)z(t) + f^*(t) + g + u(t-h), \quad t \ge 0 \\ z(t) &= \phi_z(t) = (\dot{\phi} - \dot{x}^*) - \lambda_2(t)(\phi - x^*), \quad t \in [-h, 0] \end{aligned} \tag{2.11}$$

The control $u(t-h)$ will be sought in the form

$$u(t-h) = -p\,\mathrm{sign}\,z(t-h) \tag{2.12}$$

where $p > 0$ is the control parameter. Put $\varepsilon_0 = \varepsilon/c_0$.

Theorem 1. Suppose a number $\varepsilon > 0$ is given, relations (2.5) are true, and in addition the following assumptions hold:

(1) a positive constant M exists such that

$$\|f^*(t)\| < M \tag{2.13}$$

(2) the initial function satisfies the inequality

$$\max\{|y(0)|, |z(0)|\} < \delta \tag{2.14}$$

(3) the control has the form (2.12)

(4) the quantities $\lambda_{\max}$, M and δ satisfy the inequalities

$$\begin{gathered} M < \frac{\lambda_{\max}}{2}\left(\frac{2 - e^{\lambda_{\max} h}}{e^{\lambda_{\max} h} - 1} - 2N\varepsilon_0\right)\varepsilon_0 \\ \delta < \left(e^{-\lambda_{\max} h} - \left(\frac{2M}{\lambda_{\max}\varepsilon_0} + \frac{2N\varepsilon_0}{\lambda_{\max}} + 1\right)(1 - e^{-\lambda_{\max} h})\right)\varepsilon_0 \end{gathered} \tag{2.15}$$

(5) $p = \alpha'\varepsilon_0$, where

$$\lambda_{\max} + \frac{M}{\varepsilon_0} + N\varepsilon_0 < \alpha' < \lambda_{\max}\frac{\varepsilon_0 - \delta e^{\lambda_{\max} h}}{\varepsilon_0(e^{\lambda_{\max} h} - 1)} - \frac{M}{\varepsilon_0} - N\varepsilon_0 \tag{2.16}$$

Then the solution of system (2.1) will follow the path $x^*(t)$ by means of the control (2.12), with a tracking error not exceeding ε.

Proof. We will prove that

$$|z(t)| < \varepsilon_0, \quad \forall t > 0 \tag{2.17}$$

Note that this inequality guarantees the truth of the estimate $|x(t) - x^*(t)| < c_0\varepsilon_0 = \varepsilon$ (see the proof of Lemma 1), which proves the theorem. The proof will be *reductio ad absurdum*. Suppose an instant of time $T > 0$ exists such that $|z(T)| = \varepsilon_0$. We may assume without loss of generality that $z(T) > 0$ (the proof in the case $z(T) < 0$ is analogous). Then there are two possible cases.

1. For any instant of time $0 \le t < T$, we have $0 < z(t) < \varepsilon_0$. In that case it follows from Lemma 1, inequality (2.13), Eq. (2.11), and the boundedness of $\lambda_1(t)$ that

$$\dot{z}(t) \le \lambda_{\max} z(t) + \delta_1, \quad \forall t \in [0, T]; \quad \delta_1 = M + N\varepsilon_0^2 + \alpha'\varepsilon_0$$

Using inequality (2.14), we infer from this inequality, by the Gronwall-Bellman lemma, that at time T

$$z(T) \le (\delta + \delta_1/\lambda_{\max})e^{\lambda_{\max} T} - \delta_1/\lambda_{\max}$$

Using the right-hand inequality in (2.16), we get

$$T \ge \frac{1}{\lambda_{\max}}\ln\frac{\lambda_{\max}\varepsilon_0 + \delta_1}{\delta\varepsilon_0 + \delta_1} > h$$

But this means that the following estimate is true over the interval $[h, T]$

$$\dot{z}(t) < \lambda_{\max}\varepsilon_0 + \delta_1 - 2\alpha'\varepsilon_0 < 0$$

if the left-hand inequality of (2.16) is used. Thus $z(h) < \varepsilon_0$, and the solution decreases on $[h, T]$. Hence $z(T) < \varepsilon_0$. We have obtained a contradiction.

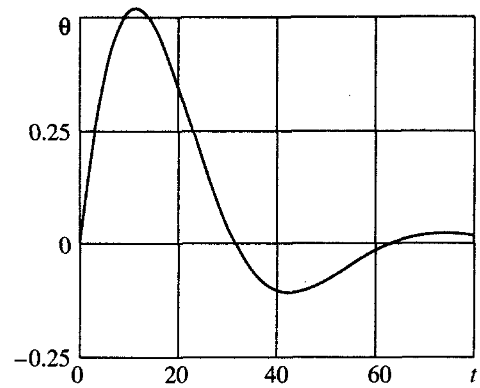

Fig. 1

2. A point $t_0 < T$ exists such that $z(t_0) = \varepsilon_0$ and $0 < z(t) < \varepsilon_0$ for $t \in (t_0, T]$. Reasoning analogous to that employed in case 1° shows that $z(T) < \varepsilon_0$. Again we have obtained a contradiction.

Thus inequality (2.17) is established.

The tracking algorithm. Summarizing what has been said up to now, we can formulate the following tracking algorithm for a second-order equation. Suppose we are given a maximum delay h in the equation, a tracked path $x^*(t)$ satisfying condition (2.13) and the first condition (2.15), and an initial function satisfying condition (2.14) and the second condition of (2.15). The tracking algorithm is a follows.

1. Fix $\varepsilon > 0$.
2. Find $\lambda_1(t)$ and $\lambda_2(t)$.
3. Choose a control parameter satisfying condition (2.16).

The tracking problem for an inverted pendulum. The equation of a controllable inverted pendulum is

$$\ddot{\theta}(t) + k\dot{\theta}(t) - p\sin\theta(t) = u(t-h), \quad p = \sqrt{g/l} \tag{2.18}$$

where θ is the angular deflection of the pendulum from the vertical axis, k is the coefficient of friction, l is the pendulum length, u is the control, and h is the delay.

Let $\theta^*(t)$ be the tracked path. Then $a^*(t) = -k$, $b^*(t) = p\cos\theta^*(t)$. The eigenvalues of the linearized system will be

$$\lambda_{1,2}(t) = (-k \pm \sqrt{k^2 + 4p\cos\theta^*(t)})/2$$

If the radicand is positive for all t, the eigenvalues of the linearized system are non-zero, simple and real, and moreover $\lambda_2(t) \leq -q < 0$.

Figure 1 illustrates the result of the simulation. The parameters are

$$\theta^*(t) = \sin(0.1t)\exp(-0.05t),\ k = 0.3,\ p = 0.04,\ h = 0.1,\ \varepsilon = 0.05,\ \alpha' = 5.6,\ \phi(t) = 0.001,$$
$$\lambda_{\max} = 0.1,\ q = 0.35$$

The norm of the tracking error is 0.013.

3. THE CONTROL OF A MECHANICAL SYSTEM

Statement of the problem. Consider a mechanical system of the form

$$H(\mathbf{q})\ddot{\mathbf{q}} + L(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{u}(t-h), \quad \mathbf{q} \in R^n \tag{3.1}$$

where $\mathbf{q}$ is the vector of generalized coordinates, $H(\mathbf{q})$ is the inertia matrix of the masses of the links, whose norm satisfies the estimate

$$1/H^+ < \|H(\mathbf{q})\| < 1/H^-, \quad \mathbf{q} \in R^n; \quad 0 < H^- \leq H^+ \tag{3.2}$$

$L(\mathbf{q}, \dot{\mathbf{q}})$ is a matrix including the relations of the forces and momenta between the masses, as well as the gravity and friction forces, $\mathbf{u}$ is the control vector, and $h > 0$ is a constant delay.

Let $\mathbf{q}^*(t) : R \to R^n$ denote the state vector of the tracked object. We shall assume that the supporting path $\mathbf{q}^*(t)$ reaches the input of the system instantaneously (or with a very small, negligible, delay). As before, we shall assume that $\mathbf{q}^*(t)$ is a slow motion.

Suppose we are given an initial function $\mathbf{\Phi}(t) = (\phi_1(t), \dots , \phi_n(t))^T$.

The tracking system may be written as

$$H(\mathbf{q})\ddot{\mathbf{q}} + L(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{u}(t-h), \quad t>0; \quad \mathbf{q}(t) = \mathbf{\Phi}(t), \quad -h \le t \le 0 \tag{3.3}$$

The problem is to find an equation of the type (1.2) guaranteeing that the solution of system (3.3) will be close to the function $\mathbf{q}^*(t)$.

Decoupling a system of n *second-order equations*. Consider the linear system

$$H\ddot{\mathbf{x}} + P\dot{\mathbf{x}} + W\mathbf{x} = 0, \quad \mathbf{x} \in R^n; \quad \det H \ne 0 \tag{3.4}$$

where H, P and W are constant $n \times n$ matrices. Multiplying both sides of the last equation on the left by H^{-1}, we obtain

$$\ddot{\mathbf{x}} + H^{-1}P\dot{\mathbf{x}} + H^{-1}W\mathbf{x} = 0 \tag{3.5}$$

Let $\mathbf{y} = (x_1, \dots , x_n, \dot{x}_1, \dots , \dot{x}_n)^T$. Transforming to a system of $2n$ first-order equations, we obtain $\dot{\mathbf{y}} = M\mathbf{y}$, where M is a $2n \times 2n$ partitioned matrix:

$$M = \begin{Vmatrix} O & I^n \\ -H^{-1}W & -H^{-1}P \end{Vmatrix} \tag{3.6}$$

where O is the $n \times n$ zero matrix and I^n is the $n \times n$ identity matrix.

Let $\sigma(M)$ denote the spectrum of the matrix M. Let $\lambda_i \in \sigma(M)$ be the real eigenvalues in the left half-plane, and let $\mathbf{h}_i$ be the corresponding eigenvectors, $i = 1, 2, \dots , n$. Write the vectors $\mathbf{h}_i$ as columns of a $2n \times n$ matrix U. We obtain

$$U = (\mathbf{h}_1, \dots, \mathbf{h}_n) = (U_1, U_2)^T$$

$$U_1 = \begin{Vmatrix} h_1^1 & h_2^1 & \dots & h_n^1 \\ \dots & \dots & \dots & \dots \\ h_1^n & h_2^n & \dots & h_n^n \end{Vmatrix}, \quad U_2 = \begin{Vmatrix} h_1^{n+1} & h_2^{n+1} & \dots & h_n^{n+1} \\ \dots & \dots & \dots & \dots \\ h_1^{2n} & h_2^{2n} & \dots & h_n^{2n} \end{Vmatrix}$$

Lemma 2. Suppose all the eigenvalues of the matrix M are simple and the matrix U_1 is invertible. Let

$$C_- = U_2U_1^{-1}, \quad C_+ = -H^{-1}P - U_2U_1^{-1} \tag{3.7}$$

Then system (3.5) may be represented as

$$(I^n d/dt - C_+)(I^n d/dt - C_-)\mathbf{x} = 0 \tag{3.8}$$

and

$$C_+ + C_- = -H^{-1}P, \quad C_+C_- = H^{-1}W \tag{3.9}$$

$$\sigma(C_+) \cup \sigma(C_-) = \sigma(M), \quad \sigma(C_-) = \{\lambda_1, \dots, \lambda_n\} \tag{3.10}$$

Proof. Writing system (3.8) in the form

$$\ddot{\mathbf{x}} - (C_+ + C_-)\dot{\mathbf{x}} + C_+C_-\mathbf{x} = 0$$

and comparing with system (3.5), we obtain relations (3.9).

It follows from these relations that $X = C_-$ satisfies the quadratic matrix equation

$$X^2 + H^{-1}PX + H^{-1}W = 0 \tag{3.11}$$

We will show that the matrix C_-, defined by the first equality of (3.7), is a solution of this equation. Let us consider (3.6) as the matrix of an operator defined in the space R^{2n}. The subspace L_n spanned by the vectors $\mathbf{h}_1, \dots, \mathbf{h}_n$ is an invariant of the operator. Then the matrix $M_L = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ satisfies the equality

$$MU = UM_L$$

from which, using the partitioned structure of the matrices M and U, we conclude that

$$U_2 = U_1M_L, \quad -H^{-1}WU_1 - H^{-1}PU_2 = U_2M_L$$

Multiplying each of these equalities on the right by U_1^{-1}, we obtain

$$C_- = U_1M_LU_1^{-1}, \quad -H^{-1}W - H^{-1}PC_- = U_2M_LU_1^{-1} \tag{3.12}$$

Multiplying the first equality of (3.12) on the left by C_- and subtracting the second from the result, we obtain (3.11) for $X = C_-$. It follows from the first relation of (3.9) that $C_+ = -H^{-1}P - U_2U_1^{-1}$.

We will now prove (3.10). Using the fact that $U_2 = U_1M_L$, we obtain $C_- = U_1M_LU_1^{-1}$. This implies the second relation of (3.10). Transforming the left-hand side of the characteristic polynomial of system (3.5)

$$|\lambda^2I^n + \lambda H^{-1}P + H^{-1}W| = |(\lambda I^n - C_+)(\lambda I^n - C_-)| = |\lambda I^n - C_+||\lambda I^n - C_-| = 0$$

we conclude that the first relation of (3.10) also holds.

Fundamental theorem. Multiplying (3.1) on the left by $H^{-1}(\mathbf{q})$ and linearizing the second term on the right of the resulting equality in the neighbourhood of $Q = (\mathbf{q}^*, \dot{\mathbf{q}}^*)$,

$$H^{-1}(\mathbf{q})L(\mathbf{q}, \dot{\mathbf{q}}) = F(\mathbf{q}, \dot{\mathbf{q}}) = (f_1(\mathbf{q}, \dot{\mathbf{q}}), \dots, f_n(\mathbf{q}, \dot{\mathbf{q}}))$$

we arrive at a system

$$\begin{aligned} &\ddot{\mathbf{q}}(t) + P(\mathbf{q}^*, \dot{\mathbf{q}}^*)(\dot{\mathbf{q}} - \dot{\mathbf{q}}^*) + W(\mathbf{q}^*, \dot{\mathbf{q}}^*)(\mathbf{q} - \mathbf{q}^*) = \\ &= -F^*(t) + g(\mathbf{q}, \mathbf{q}^*, \dot{\mathbf{q}}, \dot{\mathbf{q}}^*) + H^{-1}(\mathbf{q})\mathbf{u}(t-h), \quad t \ge 0 \\ &F^*(t) = F(\mathbf{q}^*(t), \dot{\mathbf{q}}^*(t)) \\ &P(\mathbf{q}^*, \dot{\mathbf{q}}^*) = \left\| \frac{\partial f_i}{\partial \dot{q}_j}(\mathbf{q}^*, \dot{\mathbf{q}}^*) \right\|_{i,j=1}^n, \quad W(\mathbf{q}^*, \dot{\mathbf{q}}^*) = \left\| \frac{\partial f_i}{\partial q_j}(\mathbf{q}^*, \dot{\mathbf{q}}^*) \right\|_{i,j=1}^n \end{aligned} \tag{3.13}$$

where g is the remainder term of the expansion.

Proposition 1. The matrix

$$M(t) = \left\| \begin{matrix} 0 & I^n \\ -W(\mathbf{q}^*, \dot{\mathbf{q}}^*) & -P(\mathbf{q}^*, \dot{\mathbf{q}}^*) \end{matrix} \right\|$$

is non-singular for all t.

Proposition 2. The spectrum of the matrix M consists of simple, real, non-zero eigenvalues, n of which are negative.

We can assume without loss of generality that the first n eigenvalues, say $\lambda_1(t), \dots, \lambda_n(t)$, are negative. By Lemma 2,

$$C_-(t) = U_2(t)U_1^{-1}(t), \quad C_+(t) = -P(\mathbf{q}^*, \dot{\mathbf{q}}^*) - U_2(t)U_1^{-1}(t) \tag{3.14}$$

where $U(t) = (U_1(t), U_2(t))^T$. Let $S_+(t)$ denote the matrix transforming to the Jordan basis for the matrix $C_+(t)$.

Proposition 3. For any $\varepsilon > 0$ it follows from the inequality $\|\mathbf{q}(t) - \mathbf{q}^*(t)\| < \varepsilon$ that

$$\left\|S_+^{-1}(t)H^{-1}(\mathbf{q}(t))H(\mathbf{q}^*(t))S_+(t) - I\right\| < \varepsilon, \quad t \geq 0$$

where I is the identity matrix of the appropriate order.

Lemma 3. Suppose the fundamental matrix $Y(t)$ of the homogeneous system of equations $\dot{\mathbf{y}}(t) = C_-(t)\mathbf{y}(t)$ satisfies the estimates

$$\|Y(t)\| \leq K_1 e^{-\sigma t}, \quad t \geq 0; \quad \left\|Y(t)Y^{-1}(s)\right\| \leq K_2 e^{-\rho(t-s)}, \quad t \geq s \geq 0 \tag{3.15}$$

where K_1, K_2, σ and ρ are positive constants. Then the solution of the problem

$$\dot{\mathbf{y}}(t) = C_-(t)\mathbf{y}(t) + \mathbf{r}(t); \quad \|\mathbf{r}(t)\| < \left\|S^+\right\|\varepsilon, \quad \mathbf{y}(0) = \boldsymbol{\phi}(0) - \mathbf{x}^*(0) \tag{3.16}$$

will satisfy the estimate

$$\|\mathbf{y}(t)\| < c_1\varepsilon, \quad c_1 = K_1 + K_2\|S_+\|/\rho, \quad t \geq 0 \tag{3.17}$$

Since Proposition 2 holds, system (3.13) may be transformed to

$$\begin{aligned}&(I^n d/dt - C_+(t))(I^n d/dt - C_-(t))(\mathbf{q} - \mathbf{q}^*) = \\ &= F^*(t) - \ddot{\mathbf{q}}^*(t) + g(\mathbf{q}, \mathbf{q}^*, \dot{\mathbf{q}}, \dot{\mathbf{q}}^*) - (dC_-(t)/dt)(\mathbf{q} - \mathbf{q}^*) + H^{-1}(t)\mathbf{u}(t-h)\end{aligned}$$

The matrix $C_-(t)$ has negative eigenvalues (the matrices $C_+(t)$ and $C_-(t)$ are defined in (3.14)). After the change of variables

$$\mathbf{z}(t) = S_+^{-1}(t)(I^n d/dt - C_-(t))(\mathbf{q}(t) - \mathbf{q}^*(t))$$

we obtain

$$\dot{\mathbf{z}} = J\mathbf{z} + S_+^{-1}[-F^*(t) - \ddot{\mathbf{q}}^*(t) + g_1(\mathbf{q}, \mathbf{q}^*, \dot{\mathbf{q}}, \dot{\mathbf{q}}^*) + H^{-1}\mathbf{u}(t-h)] \tag{3.18}$$

where

$$\mathbf{g}_1 = \mathbf{g} - \frac{dC_-(t)}{dt}(\mathbf{q} - \mathbf{q}^*) - \frac{dS_+(t)}{dt}\mathbf{z}$$

and $J(t) = \mathrm{diag}(\lambda_1(t), \ldots, \lambda_n(t))$ is the Jordan form of the matrix $C_+(t)$.

Lemma 4. Let $\varepsilon > 0$ be a given small number and suppose that

$$\left\|S_+^{-1}((\dot{\mathbf{q}} - \dot{\mathbf{q}}^*) - C_-(\mathbf{q} - \mathbf{q}^*))\right\| < \varepsilon$$

Then for $\|\mathbf{q} - \mathbf{q}^*\| < c_1\varepsilon$, $c_1 > 0$,

$$\left\|S_+^{-1}g_1(\mathbf{q}, \mathbf{q}^*, \dot{\mathbf{q}}, \dot{\mathbf{q}}^*)\right\| < N\varepsilon^2$$

$$N = N_1\left\|S_+^{-1}\right\|(c_1^2 + (1/\left\|S_+^{-1}\right\| + c_1\|C_-\|)^2), \quad N_1 > 0$$

The proof of Lemmas 3 and 4 is obvious, using the fact that the motion $\mathbf{q}^*$ is slow.
We now formulate the fundamental theorem.

Theorem 2. Let assumptions 1–3 hold. Suppose in addition that the following assumptions hold:
(1) the eigenvalues of the matrix $M(t)$ are bounded:

$$\|\lambda_i(t)\| \leq \lambda_{\max} \tag{3.19}$$

(2) a positive constant M^* exists such that

$$\left\| S_+^{-1}(t)(-\ddot{\mathbf{q}}^*(t) - F^*(t)) \right\| \le M^* \tag{3.20}$$

(3) the initial function satisfies the condition

$$\max\{\|\mathbf{q}(0) - \mathbf{q}^*(0)\|, \|\mathbf{z}(0)\|\} < \delta \tag{3.21}$$

(4) the control has the form

$$\mathbf{u}(t-h) = H(\mathbf{q}^*(t))S_+(t)\left\| \begin{matrix} -\alpha_1'\varepsilon\,\mathrm{sign}\,z_1(t-h) \\ \dots\dots\dots\dots\dots\dots \\ -\alpha_n'\varepsilon\,\mathrm{sign}\,z_n(t-h) \end{matrix} \right\| \tag{3.22}$$

(5) the quantities $\lambda_{\max}$, M^*, δ, α_i' satisfy inequalities

$$\lambda_{\max}h < \ln 2 \tag{3.23}$$

$$M^* < \frac{\lambda_{\max}}{2}\left(\frac{2 - e^{\lambda_{\max}h}}{e^{\lambda_{\max}h} - 1} - 2N\varepsilon\right)\varepsilon \tag{3.24}$$

$$\delta < \left(e^{-\lambda_{\max}h} - \left(\frac{2M^*}{\lambda_{\max}\varepsilon} + \frac{2N\varepsilon}{\lambda_{\max}} + 1\right)(1 - e^{-\lambda_{\max}h})\right)\varepsilon \tag{3.25}$$

$$\lambda_{\max} + \frac{M^*}{\varepsilon} + N\varepsilon < \alpha_i' + A'\varepsilon < \lambda_{\max}\frac{\varepsilon - \delta e^{\lambda_{\max}h}}{\varepsilon(e^{\lambda_{\max}h} - 1)} - \frac{M^*}{\varepsilon} - N\varepsilon \tag{3.26}$$

where

$$A' = c_1(\alpha_1' + \dots + \alpha_n'), \quad i = 1, 2, \dots, n$$

Then the solution of system (3.3) follows the path $q^*(t)$ with the help of control (3.22).

Proof. Consider system (3.18) written in coordinatewise notation

$$\dot{z}_i(t) = \lambda_i(t)z_i(t) + f_i^z(t) + r_i - \sum_{j=1}^{n} \tilde{h}_{ij}\alpha_i'\mathrm{sign}\,z_i(t-h), \quad i = 1, 2, \dots, n$$

with initial conditions

$$\mathbf{z}(t) = \boldsymbol{\phi}^z(t), \quad t \in [-h, 0]$$

where

$$\mathbf{f}^z(t) = S_+^{-1}(t)(-\mathbf{F}^*(t) - \ddot{\mathbf{q}}^*(t)), \quad \mathbf{r}(t) = S_+^{-1}\mathbf{g}_1$$

and $\tilde{h}_{ij}$ are the elements of the matrix $S_+^{-1}H^{-1}(\mathbf{q})H(\mathbf{q}^*)S_+$.
It is obvious that $\|\mathbf{z}(t)\| < \varepsilon$ for $t \ge 0$. It follows from Lemma 3 that

$$\|\mathbf{q}(t) - \mathbf{q}^*(t)\| < c_1\varepsilon, \quad t \ge 0$$

The tracking algorithm. Summarizing what has been said so far, we can formulate the following tracking algorithm for a mechanical system. Suppose the maximum delay h in the control of the system is given, as is the tracked path $\mathbf{q}^*(t)$. Let the tracked function satisfy conditions (3.20) and (3.24) and the initial function conditions (3.21) and (3.25).

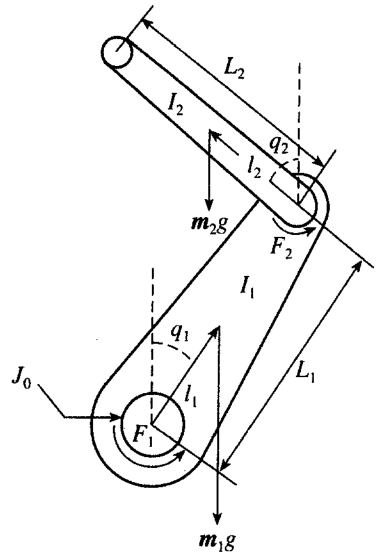

Fig. 2

The algorithm runs as follows:
1. Fix $\varepsilon > 0$.
2. Find the eigenvalues of the matrix $M(t)$.
3. Find the matrices $C_{-}(t)$ and $S_{+}(t)$.
4. Select α_i' ($i = 1, 2, \ldots, n$) in accordance with inequalities (3.26).

The tracking problem for a double inverted pendulum. The controllable motion of a double inverted pendulum (Fig. 2) is described by the following system [14].

$$H\ddot{\boldsymbol{\theta}} + P\dot{\boldsymbol{\theta}} + \mathbf{W} = \mathbf{F}; \quad \boldsymbol{\theta} = (\theta_1, \theta_2)^T, \quad \mathbf{F} = (F_1, F_2)^T$$

where

$$H = \begin{Vmatrix} J_0 + I_1 + m_1 l_1^2 + m_2 L_1^2 & m_2 L_1 l_2 \cos(\theta_1 - \theta_2) \\ m_2 L_1 l_2 \cos(\theta_1 - \theta_2) & m_2 l_2^2 + I_2 \end{Vmatrix}$$

$$P = \begin{Vmatrix} 0 & \dot{\theta}_2 m_2 L_1 l_2 \sin(\theta_1 - \theta_2) \\ -\dot{\theta}_1 m_2 L_1 l_2 \sin(\theta_1 - \theta_2) & 0 \end{Vmatrix}$$

$$W = \begin{Vmatrix} -g(m_1 l_1 + m_2 L_1)\sin\theta_1 \\ -g m_2 l_2 \sin\theta_2 \end{Vmatrix}$$

where θ_k is the angle between the vertical axis and the pendulum link, m_k is the mass of the link, L_k is the length of the link, l_k is the distance from the centre of gravity of the link to the point of support, I_k is the moment of inertia of the link, J_0 is the moment of inertia of the drive shaft, g is the acceleration due to gravity and subscript, the $k = 1$ corresponds to the lower link and $k = 2$ to the upper one.

We shall assume that the upper link also has a drive. The controls are assumed to be the torques of the drives F_1 and F_2 of the lower and upper links, respectively.

Let us assume that the maximum domain in the system is $h = 0.03$, and the tracking function is

$$\boldsymbol{\Theta}^*(t) = \begin{Vmatrix} \sin(0.1t) - 0.5 \\ \cos(0.1t)e^{-0.03t} - 0.5 \end{Vmatrix}$$

the initial function is $\boldsymbol{\Phi}(t) = \boldsymbol{\Theta}^*(t) + (0.001, 0.001)^T$.

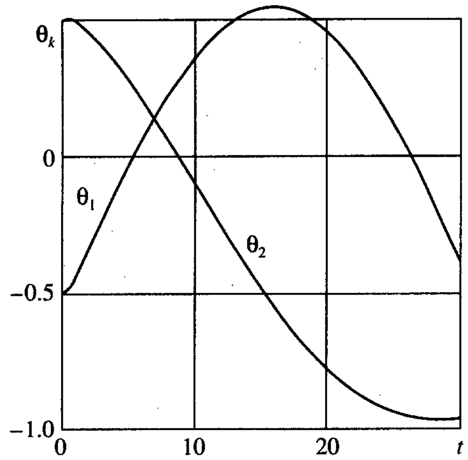

Fig. 3

The mechanical parameters of the system are assumed to be:

$m_1 = 0.132$ kg, $m_2 = 0.088$ kg, $L_1 = 0.3032$ m, $L_2 = 0.3545$ m, $l_1 = 0.1274$ m, $l_2 = 0.1209$ m, $I_1 = 0.0562$ kg $\cdot$ m^2, $I_2 = 0.0314$ kg $\cdot$ m^2, $J_0 = 6 \times 10^{-6}$ kg $\cdot$ m^2 and $g = 9.8$ m/sec^2.

The control parameters are

$$\lambda_{max} = 2.42, \quad \varepsilon = 0.05, \quad \alpha_1' = \alpha_2' = 100.$$

Figure 3 shows the results of the simulation. The norm of the tracking error for the coordinate θ_1 is 0.02, and for the coordinate θ_2, 0.025.

REFERENCES

1. YEMEL'YANOV, S. V., *Automatic Control Systems of Variable Structure*. Nedra, Moscow, 1967.
2. UTKIN, V. I., *Sliding Modes and their Application in Systems of Variable Structure*, Nauka, Moscow, 1974.
3. DOLGOLENKO, Yu. V., Sliding modes in systems of indirect on-off control. *Proc. 2nd All-Union Conf. on Automatic Control Theory*. Izd. Akad. Nauk SSSR, Moscow–Leningrad, vol. 1, pp. 421–438.
4. UTKIN, V. I., Sliding mode control design principles and applications to electric drives. *IEEE Trans. Indust. Electronics*, 1993, **40**, 1, 23–36.
5. SLOTINE, J. J. and SASTRI, S. S., Tracking control on nonlinear systems using sliding surfaces with application to robot manipulators. *Intern. J. Control*, 1983, **38**, 2, 465–492.
6. STRYGIN, V. V., FRIDMAN, L. M. and POLYAKOV, A. Ye., Local stabilization of on-off systems with delay. *Dokl. Ross. Akad. Nauk*, 2001, **379**, 5, 603–605.
7. MATYUKHIN, V. I. and PYATNITSKII, Ye. S., Control of the motion of robot manipulators based on the decomposition principle taking the dynamics of the drives into account. *Avtomatika i Telemekhanika*, 1989, 9, 67–81.
8. MUKHAMETZYANOV, I. A., The design of systems with asymptotically stable programmed constraints. *Prikl. Mat. Mekh.*, 2001, **65**, 5, 822–830.
9. CHERNOUS'KO, F. L., The controllable motions of a two-link mechanism along a horizontal plane. *Prikl. Mat. Mekh.*, 2001, **65**, 4, 578–591.
10. MANITA, L. A., Optimal modes with chattering switchings in manipulator control problems. *Prikl. Mat. Mekh.*, 2000, **64**, 1, 19–28.
11. UTKIN, V. A., Control Problems for an asynchronous electric drive. *Avtomatika i Telemekhanika*, 1993, **12**, 53–56.
12. MEDVEDEV, V. S., LESKOV, A. G. and YUSHCHENKO, A. S., *Control Systems of Robot Manipulators*. Nauka, Moscow, 1978.
13. IKRAMOV, Kh. D., *The numerical Solution of Matrix Equations*. Nauka, Moscow, 1984.
14. LEDGERWOOD, T. and MISAWA, E., Controllability and nonlinear control of the rotation of an inverted pendulum. In *ASME Winter Ann. Meeting* (Edited by E. A. Misawa), Anaheim, CA, 1992, Vol. 43, 81–88.

Translated by D.L.